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A RIEMANN SOLVER FOR THE TWO-DIMENSIONAL MHD EQUATIONS

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SUMMARY

This paper presents how the equations of magnetohydrodynamics (MHD) in primitive form should be written in conservative form with the inclusion of a divergence source along with a divergence wave and how a physically correct sonic fix can be embedded directly in the fluxes. The numerical scheme was applied to a blast wave problem in which a circular energetic plasma is released in a free and magnetized medium with a reflected wall. The results show that the method with the new sonic fix can handle the divergence condition on the magnetic field and produces an almost uniform shock compression in all directions, resolving the shocks and discontinuities rather sharply. © 1997 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The equations of magnetohydrodynamics (MHD) describe the transients and equilibrium states of the plasma particles[1] that interact with electric and magnetic fields by means of the plasma waves whose properties can be analytically investigated.[2,3] In this work, a Roe-type approximate Riemann solver[4] is developed for the ideal MHD equations with a new Roe averaging,[5] a new sonic fix,[6] a divergence source and a divergence wave which completes the eight-wave structure required to remove the numerical errors arising from the divergence condition on the magnetic field.

2. MHD EQUATIONS WITH A $\vec{\nabla}\cdot\vec{B}$ SOURCE AND A NEW SONIC FIX

Preserving the condition $\vec{\nabla}\cdot\vec{B}=0$ to highest numerical accuracy in solving the conservative form of the MHD equations has been a very difficult problem.[7] Recently it was shown that, when the primitive form of the multidimensional MHD equations is carefully analysed, one ends up with a divergence wave along with seven others and source terms related to $\vec{\nabla}\cdot\vec{B}$[1,8] in the conservative form of the momentum equation and Faraday's law. While the magnitude of this source is very small, it

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has the stabilizing effects against the magnetic monopoles possibly created by numerics due to the truncation and/or systematic errors (see the discussions given in Reference 9). The conservative form of the MHD equations including the divergence source is given by

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\vec{V}\\ \vec{B}\\ E\end{bmatrix}+\vec{\nabla}\cdot\begin{bmatrix}\rho\vec{V}\\ \rho\vec{V}\vec{V}+P+B^2/8\pi-\vec{B}\vec{B}/4\pi\\ \vec{V}\vec{B}-\vec{B}\vec{V}\\ (E+P+B^2/8\pi)\vec{V}-(\vec{B}/4\pi)\vec{B}\cdot\vec{V}\end{bmatrix}=-\begin{bmatrix}0\\ \vec{B}/4\pi\\ \vec{V}\\ \vec{B}\cdot\vec{V}/4\pi\end{bmatrix}\vec{\nabla}\cdot\vec{B}, \tag{1}$$

where $\rho, P, \vec{V}$ and $\vec{B}$ are the density, pressure, velocity and magnetic field respectively and $E=\frac{1}{2}\rho V^2+P/(\gamma-1)+B^2/8\pi$ is the total energy, with γ the ratio of specific heats.

The differential form of the MHD equations (1) in two dimensions represents the conservation of mass, momentum, magnetic and energy fluxes on the x–y plane with a symmetry given by $\partial/\partial z=0$. When these equations are integrated over a finite volume ($\mathrm{d}V=\mathrm{d}A\,\mathrm{d}t$), one gets the integral and discrete forms

$$\iiint_V \vec{U}_t\,\mathrm{d}A\,\mathrm{d}t+\iiint_V(\vec{F}_x+\vec{G}_y)\,\mathrm{d}A\,\mathrm{d}t=\iiint_V\vec{S}\,\mathrm{d}A\,\mathrm{d}t, \tag{2}$$

$$\langle\vec{U}\rangle^{n+1}=\langle\vec{U}\rangle^{n}-\frac{\Delta t}{A}\sum_{k=1}^{4\text{ or }3}(\vec{F}_n^k)^{n+1/2}\Delta S_k+\Delta t\langle\vec{S}\rangle^{n+1/2} \tag{3}$$

by employing the Gauss theorem. Here $\vec{U}$ is the state vector, A is the area of the cell whose kth cell face length is ΔS_k and $\vec{F}_n^k$ is the familiar numerical flux given by

$$\vec{F}(U_\mathrm{L},U_\mathrm{R})=\tfrac{1}{2}[\vec{F}(U_\mathrm{L})+\vec{F}(U_\mathrm{R})]-\tfrac{1}{2}\sum_{k=1}^{8}|\tilde{\lambda}_k|\tilde{\alpha}_k\vec{r}_k, \tag{4}$$

where $\tilde{\lambda}_k$ and $\vec{r}_k$ are the kth eigenvalue and right eigenvector of the flux Jacobian $A(\tilde{U})$ respectively and $\tilde{\alpha}_k$ is the strength of the kth wave, all evaluated at the interfaces as a function of the averaged primitive state $\vec{W}=[\tilde{\rho},\tilde{V}_x,\tilde{V}_y,\tilde{V}_z,\tilde{B}_x,\tilde{B}_y,\tilde{B}_z,\tilde{P}]^\mathrm{T}$.

The flux given by (4) produces unphysical expansion shock at sonic point (at which at least one λ_k vanishes and satisfies $\lambda_k^\mathrm{R}>0$, $\lambda_k^\mathrm{L}<0$), since the dissipation sum associated with this wave vanishes. In order to satisfy the entropy condition, one has to build into the flux a model of a smooth transonic expansion wave. This can be done by applying a pointwise dissipation by modifying $|\lambda_k|$ on an empirical level (see Reference 10 as an example). A more physical mechanism presented recently by Aslan and Kammash[11,12] was proven to have created an accurate and robust fix. The procedure will not be repeated here, but the original feature of this fix is that a sonic dissipation term is directly embedded in the fluxes, to be used only at sonic points. Considering this new sonic fix, the modified normal flux is now given by

$$\vec{F}(U_\mathrm{L},U_\mathrm{R})=(I-K^*)\frac{\vec{F}_\mathrm{L}+\vec{F}_\mathrm{R}}{2}-\tfrac{1}{2}\sum_{k=1}^{8}(1-\tilde{\kappa}_k^*)|\tilde{\lambda}_k|\tilde{\alpha}_k\tilde{\vec{r}}_k, \tag{5}$$

where $\tilde{\kappa}_k^*=(\Delta t/\Delta x)(\lambda_\mathrm{R}-\lambda_\mathrm{L})/2$ and $K^*=(\Delta t/\Delta x)(A_\mathrm{R}-A_\mathrm{L})/2$ are called the sonic fix parameter and sonic fix matrix respectively. This flux can easily be improved to second order by means of flux limiters (see Reference 11 for details).

3. NUMERICAL RESULTS

The scheme described above has been tested with an instantaneous high-beta [$\beta = P/(B^2/8\pi)$] blast in which an energetic plasma propagates in a free and magnetized space above a reflective lower boundary. The blast wave is driven by a circular region ($r = 0{\cdot}24$) with a large overpressure and density. The initial conditions inside and outside the region are given by

$$W_{\text{in}} = [20, 0, 0, 0, B_x^0, B_y^0, 0, 50]^{\text{T}}, \qquad W_{\text{out}} = [1, 0, 0, 0, B_x^0, B_y^0, 0, 1]^{\text{T}}, \tag{6}$$

with $\lambda = 1{\cdot}4$. When this problem is solved with zero initial magnetic field, the fluid particles propagate symmetrically outside the blast centre. However, the existence of a strong magnetic field gives rise to anisotropy in the density and pressure during propagation. The test case has been run on an 80×80 Cartesian grid with $x = [-1, 1], y = [-0{\cdot}8, 1{\cdot}2]$ and the solutions are obtained at time levels of 50, 100, 150 and 200 with a safety factor of 0·4. Figure 1 shows the time evolution of pressure contours obtained with initial magnetic fields of $B_x^0 = B_y^0 = 0$ (Figure 1(b)) and $B_x^0 = 8, B_y^0 = 2$ (Figure 1(a)). With zero magnetic field, the energy in the central region creates a shock wave followed by a weak contact discontinuity, both of which propagate symmetrically outwards. When the shock reflects off the lower surface (at time step 150), the surface pressure (and hence the temperature) increases and the reflected shock interacts with the contact discontinuity. During this time the density and pressure are both reduced to lower levels in the central region.

When a relatively strong magnetic field exists in the medium, the central region is tilted and elongated along the direction in which the magnetic field strength is maximum. This is depicted in Figure 1(a) by the time evolution of pressure contours. How the flow and magnetic field configurations evolve in time is depicted in Figure 2. It is remarked that the code crashes nearly after time step 100 when no divergence source (along with the divergence wave) is used. This shows the importance of divergence source stabilization in finite volume schemes. However, the source is not needed when a fluctuation approach is used as will be shown in subsequent papers.

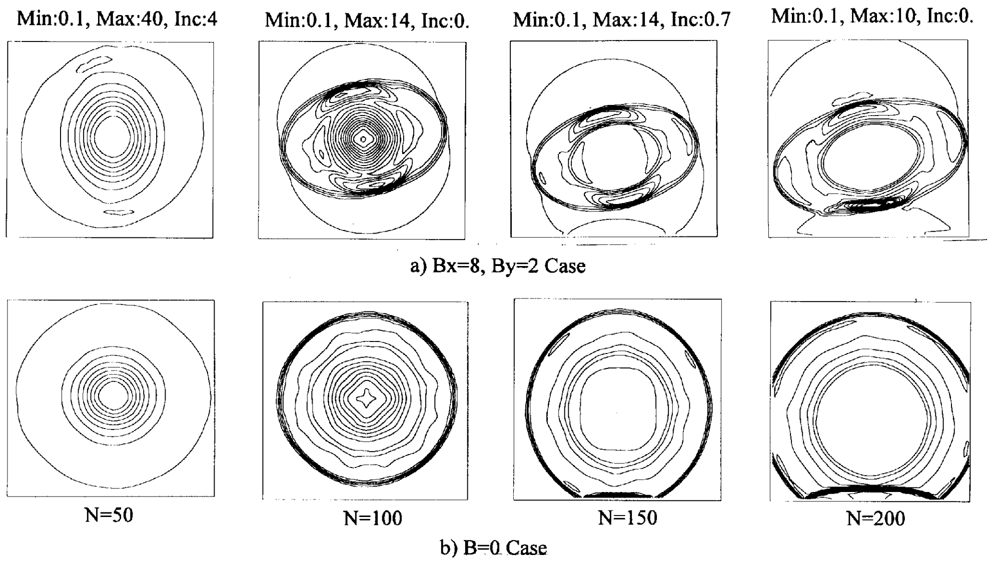

Figure 1. Circular explosion problem: pressure contours obtained with (a) initial magnetic field of $B_x^0 = 8, B_y^0 = 2$ and (b) zero initial magnetic field

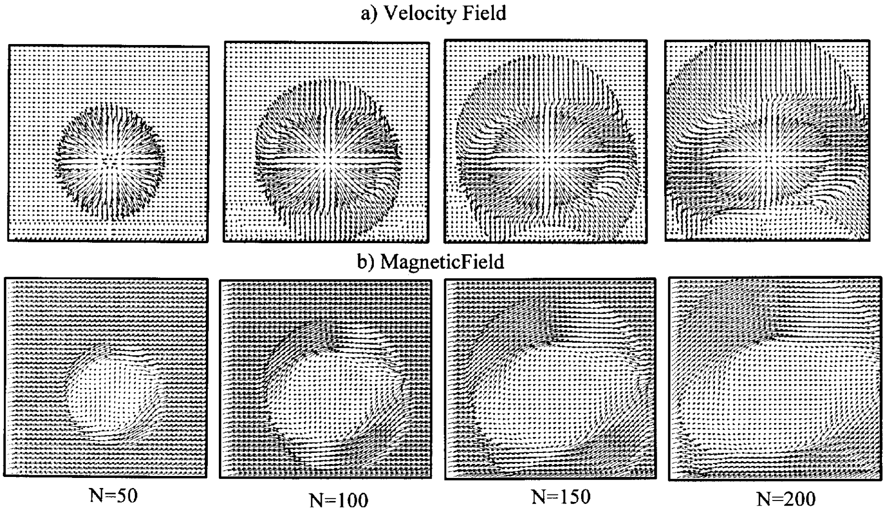

Figure 2. Circular explosion problem: vector plots of (a) velocity and (b) magnetic fields showing propagation direction and magnetic field configuration. The results were obtained with an initial magnetic field $B_x^0 = 8, B_y^0 = 2$

A close examination of these results shows that the shock profiles preserve the same amount of sharpness regardless of direction even though no rotation is applied. Also, the new sonic fix seems to have produced no problems and worked very well during the iterations. These results show that the code is capable of producing physically correct solutions on Cartesian grids (and quadrilateral grids[6]).

4. CONCLUSIONS

The MHD solver described here handles the sonic points and the errors made in preserving the divergence condition and leads to uniform compression ratios in all directions even though no flux rotation and mesh refinement are utilized. The success of this code lies behind the facts that the new sonic fix is embedded directly into the fluxes, flux limiters are used instead of state limiters, the telescoping property is preserved very carefully and, most importantly, the divergence condition is preserved by means of a source and a new divergence wave. The next objective is to use quadrilateral or triangular cells to simulate plasma equilibria in cylindrical, spherical, and toroidal geometries.

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